UNIT II

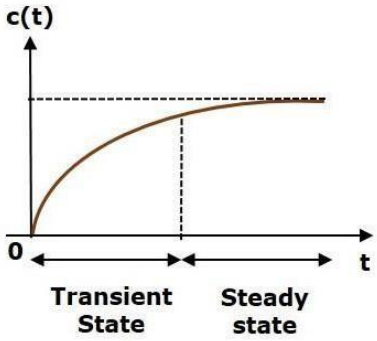
TIME – DOMAIN ANALYSIS WITH INPUT-OUTPUT MODELS

What is Time Response?

If the output of control system for an input varies with respect to time, then it is called the **time response** of the control system. The time response consists of two parts.

- Transient response
- Steady state response

The response of control system in time domain is shown in the following figure.



Here, both the transient and the steady states are indicated in the figure. The responses corresponding to these states are known as transient and steady state responses.

Where,

- $c_{tr}(t)$ is the transient response
- $c_{ss}(t)$ is the steady state response

Transient Response

After applying input to the control system, output takes certain time to reach steady state. So, the output will be in transient state till it goes to a steady state. Therefore, the response of the control system during the transient state is known as **transient response**.

The transient response will be zero for large values of 't'. Ideally, this value of 't' is infinity and practically, it is five times constant.

Mathematically, we can write it as

$$\lim_{t\to\infty} c_{tr}(t)=0 \lim_{t\to\infty} c_{tr}(t)=0$$

Steady state Response

The part of the time response that remains even after the transient response has zero value for large values of 't' is known as **steady state response**. This means, the transient response will be zero even during the steady state.

STANDARD TEST SIGNALS

The standard test signals are impulse, step, ramp and parabolic. These signals are used to know the performance of the control systems using time response of the output.

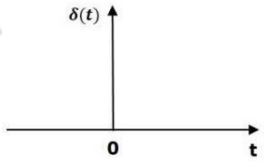
Unit Impulse Signal

A unit impulse signal, $\delta(t)$ is defined as

$$\delta(t)=0 \delta(t)=0 \text{ for } t\neq 0 t\neq 0$$

$$\text{and } \int_{0+0-}\delta(t)dt=1 \int_{0-0+}\delta(t)dt=1$$

The following figure shows unit impulse signal.



So, the unit impulse signal exists only at 't' is equal to zero. The area of this signal under small interval of time around 't' is equal to zero is one. The value of unit impulse signal is zero for all other values of 't'.

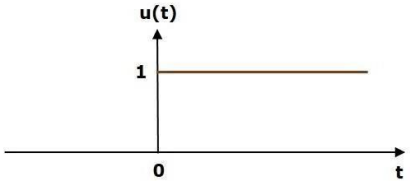
Unit Step Signal

A unit step signal, u(t) is defined as

$$u(t)=1;t\geq 0 u(t)=1;t\geq 0$$

=0;t<0=0;t<0

Following figure shows unit step signal.



So, the unit step signal exists for all positive values of 't' including zero. And its value is one during this interval. The value of the unit step signal is zero for all negative values of 't'.

Unit Ramp Signal

A unit ramp signal, r(t) is defined as

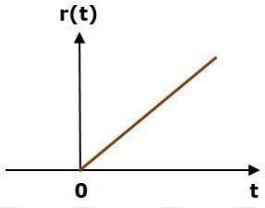
$$r(t)=t;t\geq 0$$

=0;t<0=0;t<0

We can write unit ramp signal, r(t)r(t) in terms of unit step signal, u(t)u(t) as

$$r(t)=tu(t)$$

Following figure shows unit ramp signal.



So, the unit ramp signal exists for all positive values of 't' including zero. And its value increases linearly with respect to 't' during this interval. The value of unit ramp signal is zero for all negative values of 't'.

Unit Parabolic Signal

A unit parabolic signal, p(t) is defined as,

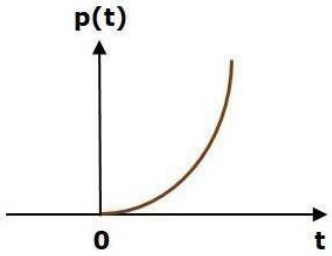
$$p(t)=t^2/2;t\geq 0$$

=0;t<0=0;t<0

We can write unit parabolic signal, p(t)p(t) in terms of the unit step signal, u(t)u(t)as,

$$p(t)=t^2/2u(t)$$

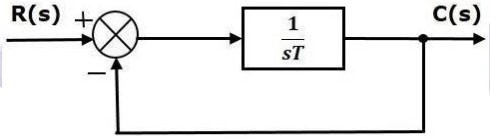
The following figure shows the unit parabolic signal.



So, the unit parabolic signal exists for all the positive values of **'t'** including zero. And its value increases non-linearly with respect to 't' during this interval. The value of the unit parabolic signal is zero for all the negative values of 't'.

Response of the First Order System

In this chapter, let us discuss the time response of the first order system. Consider the following block diagram of the closed loop control system. Here, an open loop transfer function, 1sT1sT is connected with a unity negative feedback.



We know that the transfer function of the closed loop control system has unity negative feedback as,

$$C(s)R(s)=G(s)1+G(s)C(s)R(s)=G(s)1+G(s)$$

Substitute, $G(s)=1sTG(s)=1sT$ in the above equation.

$$C(s)R(s)=1sT1+1sT=1sT+1C(s)R(s)=1sT1+1sT=1sT+1$$

The power of s is one in the denominator term. Hence, the above transfer function is of the first order and the system is said to be the **first order system**.

We can re-write the above equation as

$$C(s)=(1sT+1)R(s)C(s)=(1sT+1)R(s)$$

Where,

- **C(s)** is the Laplace transform of the output signal c(t),
- **R(s)** is the Laplace transform of the input signal r(t), and
- **T** is the time constant.

Follow these steps to get the response (output) of the first order system in the time domain.

- Take the Laplace transform of the input signal $r(t)$r(t).
- Consider the equation, $C(s)=(1sT+1)R(s)C(s)=(1sT+1)R(s)$
- Substitute $R(s)$R(s) value in the above equation.
- Do partial fractions of $C(s)$C(s) if required.
- Apply inverse Laplace transform to $C(s)$C(s).

In the previous chapter, we have seen the standard test signals like impulse, step, ramp and parabolic. Let us now find out the responses of the first order system for each input, one by one. The name of the response is given as per the name of the input signal. For example, the response of the system for an impulse input is called as impulse response.

Step Response of First Order System

Consider the **unit step signal** as an input to first order system.

So, $r(t)=u(t)$r(t)=u(t)

Apply Laplace transform on both the sides.

$$R(s)=1sR(s)=1s$$

Consider the equation, $C(s)=(1sT+1)R(s)$C(s)=(1sT+1)R(s)

Substitute, $R(s)=1sR(s)=1s$ in the above equation.

$$C(s)=(1sT+1)(1s)=1s(sT+1)C(s)=(1sT+1)(1s)=1s(sT+1)$$

Do partial fractions of C(s).

$$C(s)=1s(sT+1)=As+BsT+1C(s)=1s(sT+1)=As+BsT+1$$

$$\Rightarrow 1s(sT+1)=A(sT+1)+Bss(sT+1)\Rightarrow 1s(sT+1)=A(sT+1)+Bss(sT+1)$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$1=A(sT+1)+Bs1=A(sT+1)+Bs$$

By equating the constant terms on both the sides, you will get A = 1.

Substitute, A = 1 and equate the coefficient of the **s** terms on both the sides.

$$0=T+B\Rightarrow B=-T0=T+B\Rightarrow B=-T$$

Substitute, A = 1 and B = −T in partial fraction expansion of $C(s)$C(s).

$$C(s)=1s-TsT+1=1s-TT(s+1T)C(s)=1s-TsT+1=1s-TT(s+1T)$$

$$\Rightarrow C(s)=1s-1s+1T\Rightarrow C(s)=1s-1s+1T$$

Apply inverse Laplace transform on both the sides.

$$c(t)=(1-e-(tT))u(t)c(t)=(1-e-(tT))u(t)$$

The **unit step response**, c(t) has both the transient and the steady state terms.

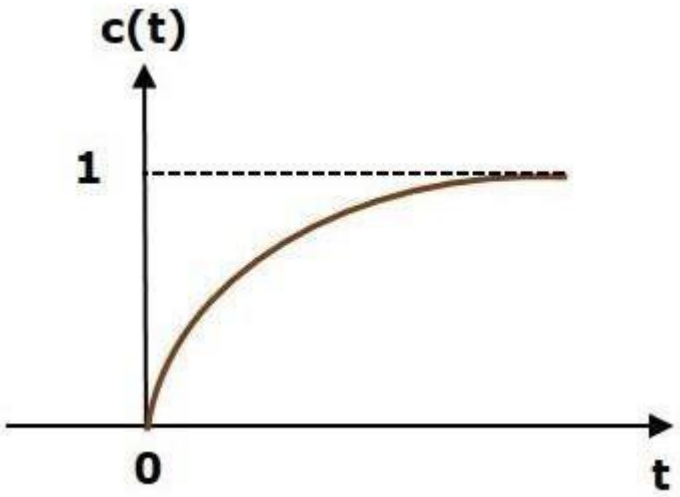
The transient term in the unit step response is -

$$ctr(t)=-e-(tT)u(t)ctr(t)=-e-(tT)u(t)$$

The steady state term in the unit step response is -

$$css(t)=u(t)css(t)=u(t)$$

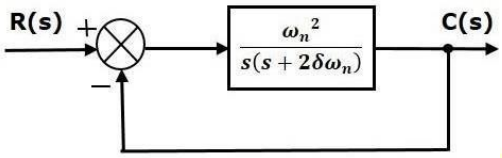
The following figure shows the unit step response.



The value of the **unit step response, c(t)** is zero at t = 0 and for all negative values of t. It is gradually increasing from zero value and finally reaches to one in steady state. So, the steady state value depends on the magnitude of the input.

Response of Second Order System

In this chapter, let us discuss the time response of second order system. Consider the following block diagram of closed loop control system. Here, an open loop transfer function, ω2ns(s+2δωn)ωn2s(s+2δωn) is connected with a unity negative feedback.



We know that the transfer function of the closed loop control system having unity negative feedback as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\frac{C(s)}{R(s)} = \frac{\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)}{1+\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)} = \frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2}$$

The power of ‘s’ is two in the denominator term. Hence, the above transfer function is of the second order and the system is said to be the **second order system**.

The characteristic equation is $s^2+2\delta\omega_n s+\omega_n^2=0$

$$\Rightarrow s = -\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1}$$

The roots of characteristic equation are -

- The two roots are imaginary when δ = 0.
- The two roots are real and equal when δ = 1.
- The two roots are real but not equal when δ > 1.
- The two roots are complex conjugate when 0 < δ < 1.

Step Response of Second Order System

Consider the unit step signal as an input to the second order system.

Laplace transform of the unit step signal is,

$$R(s) = \frac{1}{s}$$

We know the transfer function of the second order closed loop control system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Case 1: δ = 0

Substitute, $\delta = 0$ in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{s^2+\omega_n^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s^2+\omega_n^2)}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1-\cos(\omega_n t))\, u(t)$$

So, the unit step response of the second order system when $/delta = 0$ will be a continuous time signal with constant amplitude and frequency.

Case 2: δ = 1

Substitute, $/delta = 1$ in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\omega_n s+\omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s+\omega_n)^2}\right) R(s)$$

$$C(s) = \left(\frac{\omega_n^2}{(s+\omega_n)^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

After simplifying, you will get the values of A, B and C as $1,\ -1\ and\ -\omega_n$ respectively. Substitute these values in the above partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t})u(t)$$

So, the unit step response of the second order system will try to reach the step input in steady state.

Case 3: 0 < δ < 1

We can modify the denominator term of the transfer function as follows –

$$s^2+2\delta\omega_n s+\omega_n^2=\left\{s^2+2(s)(\delta\omega_n)+(\delta\omega_n)^2\right\}+\omega_n^2-(\delta\omega_n)^2$$

$$=(s+\delta\omega_n)^2+\omega_n^2(1-\delta^2)$$

The transfer function becomes,

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{(s+\delta\omega_n)^2+\omega_n^2(1-\delta^2)}$$

$$\Rightarrow C(s)=\left(\frac{\omega_n^2}{(s+\delta\omega_n)^2+\omega_n^2(1-\delta^2)}\right)R(s)$$

Substitute, $R(s)=\frac{1}{s}$ in the above equation.

$$C(s)=\left(\frac{\omega_n^2}{(s+\delta\omega_n)^2+\omega_n^2(1-\delta^2)}\right)\left(\frac{1}{s}\right)=\frac{\omega_n^2}{s\left((s+\delta\omega_n)^2+\omega_n^2(1-\delta^2)\right)}$$

Do partial fractions of $C(s)$.

$$C(s)=\frac{\omega_n^2}{s\left((s+\delta\omega_n)^2+\omega_n^2(1-\delta^2)\right)}=\frac{A}{s}+\frac{Bs+C}{(s+\delta\omega_n)^2+\omega_n^2(1-\delta^2)}$$

After simplifying, you will get the values of A, B and C as $1,\ -1\ and\ -2\delta\omega_n$ respectively. Substitute these values in the above partial fraction expansion of C(s).

$$C(s)=\frac{1}{s}-\frac{s+2\delta\omega_n}{(s+\delta\omega_n)^2+\omega_n^2(1-\delta^2)}$$

$$C(s)=\frac{1}{s}-\frac{s+\delta\omega_n}{(s+\delta\omega_n)^2+\omega_n^2(1-\delta^2)}-\frac{\delta\omega_n}{(s+\delta\omega_n)^2+\omega_n^2(1-\delta^2)}$$

$$C(s)=\frac{1}{s}-\frac{(s+\delta\omega_n)}{(s+\delta\omega_n)^2+(\omega_n\sqrt{1-\delta^2})^2}-\frac{\delta}{\sqrt{1-\delta^2}}\left(\frac{\omega_n\sqrt{1-\delta^2}}{(s+\delta\omega_n)^2+(\omega_n\sqrt{1-\delta^2})^2}\right)$$

Substitute, $\omega_n\sqrt{1-\delta^2}$ as ω_d in the above equation.

$$c(t) = \left(1 - e^{-\delta\omega_n t}\cos(\omega_d t) - \frac{\delta}{\sqrt{1-\delta^2}} e^{-\delta\omega_n t}\sin(\omega_d t)\right) u(t)$$

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\left((\sqrt{1-\delta^2})\cos(\omega_d t) + \delta\sin(\omega_d t)\right)\right) u(t)$$

If $\sqrt{1-\delta^2} = \sin(\theta)$, then 'δ' will be cos(θ). Substitute these values in the above equation.

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}(\sin(\theta)\cos(\omega_d t) + \cos(\theta)\sin(\omega_d t))\right) u(t)$$

$$\Rightarrow c(t) = \left(1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)\right) u(t)$$

So, the unit step response of the second order system is having damped oscillations (decreasing amplitude) when 'δ' lies between zero and one.

Case 4: δ > 1

We can modify the denominator term of the transfer function as follows –

$$s^2 + 2\delta\omega_n s + \omega_n^2 = \{s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2\} + \omega_n^2 - (\delta\omega_n)^2$$

$$= (s + \delta\omega_n)^2 - \omega_n^2\left(\delta^2 - 1\right)$$

The transfer function becomes,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\delta\omega_n)^2 - \omega_n^2(\delta^2-1)}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 - \omega_n^2(\delta^2-1)}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 - (\omega_n\sqrt{\delta^2-1})^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s+\delta\omega_n+\omega_n\sqrt{\delta^2-1})(s+\delta\omega_n-\omega_n\sqrt{\delta^2-1})}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s(s+\delta\omega_n+\omega_n\sqrt{\delta^2-1})(s+\delta\omega_n-\omega_n\sqrt{\delta^2-1})}$$

$$= \frac{A}{s} + \frac{B}{s+\delta\omega_n+\omega_n\sqrt{\delta^2-1}} + \frac{C}{s+\delta\omega_n-\omega_n\sqrt{\delta^2-1}}$$

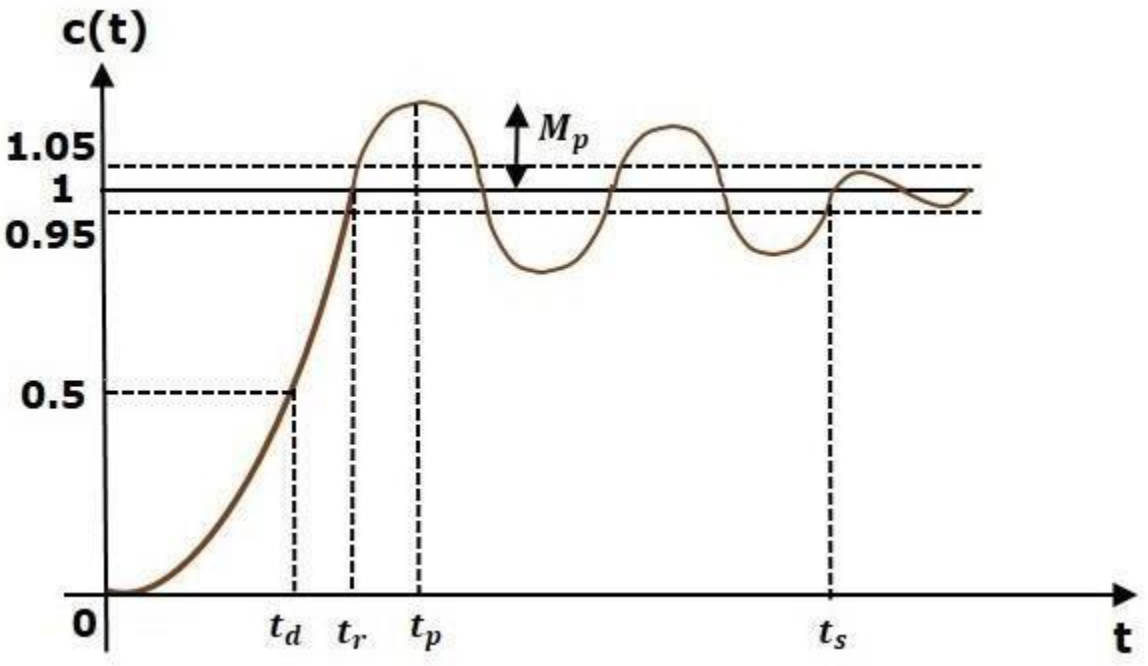
Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 + \left(\frac{1}{2(\delta+\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\right) e^{-(\delta\omega_n+\omega_n\sqrt{\delta^2-1})t} - \left(\frac{1}{2(\delta-\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\right) e^{-(\delta\omega_n-\omega_n\sqrt{\delta^2-1})t}\right) u(t)$$

Since it is over damped, the unit step response of the second order system when δ > 1 will never reach step input in the steady state.

Time Domain Specifications

In this chapter, let us discuss the time domain specifications of the second order system. The step response of the second order system for the underdamped case is shown in the following figure.



All the time domain specifications are represented in this figure. The response up to the settling time is known as transient response and the response after the settling time is known as steady state response.

Delay Time

It is the time required for the response to reach **half of its final value** from the zero instant. It is denoted by ŧdtd.

Consider the step response of the second order system for t ≥ 0, when 'δ' lies between zero and one.

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right) \sin(\omega_d t + \theta)$$

The final value of the step response is one.

Therefore, at $t=t_{dt=td}$, the value of the step response will be 0.5. Substitute, these values in the above equation.

$$c(t_d) = 0.5 = 1 - \left(\frac{e^{-\delta\omega_n t_d}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_d + \theta)$$

$$\Rightarrow \left(\frac{e^{-\delta\omega_n t_d}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_d + \theta) = 0.5$$

By using linear approximation, you will get the **delay time t_d** as

$$t_d = \frac{1+0.7\delta}{\omega_n}$$

Rise Time

It is the time required for the response to rise from **0% to 100% of its final value**. This is applicable for the **under-damped systems**. For the over-damped systems, consider the duration from 10% to 90% of the final value. Rise time is denoted by **t_r**.

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

$$c(t_2) = 1 = 1 - \left(\frac{e^{-\delta\omega_n t_2}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_2 + \theta)$$

$$\Rightarrow \left(\frac{e^{-\delta\omega_n t_2}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_2 + \theta) = 0$$

$$\Rightarrow \sin(\omega_d t_2 + \theta) = 0$$

$$\Rightarrow \omega_d t_2 + \theta = \pi$$

$$\Rightarrow t_2 = \frac{\pi - \theta}{\omega_d}$$

Substitute t_1 and t_2 values in the following equation of **rise time**,

$$t_r = t_2 - t_1$$

$$\therefore t_r = \frac{\pi - \theta}{\omega_d}$$

From above equation, we can conclude that the rise time t_{rtr} and the damped frequency $\omega_{d\omega d}$ are inversely proportional to each other.

Peak Time

It is the time required for the response to reach the **peak value** for the first time. It is denoted by t_{ptp}. At $t=t_{pt=tp}$, the first derivate of the response is zero.

We know the step response of second order system for under-damped case is

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \right) \sin(\omega_d t + \theta)$$

Differentiate $c(t)$ with respect to 't'.

$$\frac{\mathrm{d}c(t)}{\mathrm{d}t} = -\left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \right) \omega_d \cos(\omega_d t + \theta) - \left(\frac{-\delta\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \right) \sin(\omega_d t + \theta)$$

Substitute, $t = t_p$ and $\frac{\mathrm{d}c(t)}{\mathrm{d}t} = 0$ in the above equation.

$$0 = -\left(\frac{e^{-\delta\omega_n t_p}}{\sqrt{1-\delta^2}} \right) [\omega_d \cos(\omega_d t_p + \theta) - \delta\omega_n \sin(\omega_d t_p + \theta)]$$

$$\Rightarrow \omega_n \sqrt{1-\delta^2} \cos(\omega_d t_p + \theta) - \delta\omega_n \sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sqrt{1-\delta^2} \cos(\omega_d t_p + \theta) - \delta \sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sin(\theta) \cos(\omega_d t_p + \theta) - \cos(\theta) \sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sin(\theta - \omega_d t_p - \theta) = 0$$

$$\Rightarrow sin(-\omega_d t_p) = 0 \Rightarrow -\sin(\omega_d t_p) = 0 \Rightarrow sin(\omega_d t_p) = 0$$

$$\Rightarrow \omega_d t_p = \pi$$

$$\Rightarrow t_p = \frac{\pi}{\omega_d}$$

From the above equation, we can conclude that the peak time t_p and the damped frequency ω_d are inversely proportional to each other.

Peak Overshoot

Peak overshoot $\mathbf{M_p}$ is defined as the deviation of the response at peak time from the final value of response. It is also called the **maximum overshoot**.

Mathematically, we can write it as

$$M_p = c(t_p) - c(\infty)$$

Where,

$c(t_p)$ is the peak value of the response.

$c(\infty)$ is the final (steady state) value of the response.

$$c(t_p) = 1 - \left(\frac{e^{-\delta\omega_n t_p}}{\sqrt{1-\delta^2}} \right) \sin(\omega_d t_p + \theta)$$

Substitute, $t_p = \frac{\pi}{\omega_d}$ in the right hand side of the above equation.

$$c(t_P) = 1 - \left(\frac{e^{-\delta\omega_n \left(\frac{\pi}{\omega_d}\right)}}{\sqrt{1-\delta^2}} \right) \sin\left(\omega_d \left(\frac{\pi}{\omega_d} \right) + \theta \right)$$

$$\Rightarrow c(t_p) = 1 - \left(\frac{e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}}{\sqrt{1-\delta^2}} \right) (-\sin(\theta))$$

We know that

$$\sin(\theta) = \sqrt{1-\delta^2}$$

So, we will get $c(t_p)$ as

$$c(t_p) = 1 + e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}$$

Settling time

It is the time required for the response to reach the steady state and stay within the specified tolerance bands around the final value. In general, the tolerance bands are 2% and 5%. The settling time is denoted by t_{sts}.

The settling time for 5% tolerance band is -

$$t_s = \frac{3}{\delta\omega_n} = 3\tau$$

The settling time for 2% tolerance band is -

$$t_s = \frac{4}{\delta\omega_n} = 4\tau$$

Where, τ is the time constant and is equal to $\frac{1}{\delta\omega_n}$.

Time domain specification	Formula	Substitution of values in Formula	Final value
Delay time	$t_d=1+0.7\delta\omega_n t_d=1+0.7\delta\omega n$	$t_d=1+0.7(0.5)2t_d=1+0.7(0.5)2$	$t_{dtd}=0.675$ sec
Rise time	$t_r=\pi-\theta\omega_d t_r=\pi-\theta\omega d$	$t_r=\pi-(\pi_3)1.732t_r=\pi-(\pi3)1.732$	$t_{rtr}=1.207$ sec
Peak time	$t_p=\pi\omega_d t_p=\pi\omega d$	$t_p=\pi1.732t_p=\pi1.732$	$t_{ptp}=1.813$ sec
% Peak overshoot	$\%M_p=\lvert e^{-(\delta\pi1-\delta2\sqrt)}\rvert\times100$ $\%\%Mp=(e-(\delta\pi1-\delta2))\times100\%$	$\%M_p=\lvert e^{-(0.5\pi1-(0.5)2\sqrt)}\rvert\times100\%$ $\%Mp=(e-(0.5\pi1-(0.5)2))\times100\%$	$\%M_p\%Mp=16.32\%$
Settling time for 2% tolerance band	$t_s=4\delta\omega_n t_s=4\delta\omega n$	$t_S=4(0.5)(2)t_S=4(0.5)(2)$	$t_{sts}=4$ sec

Control Systems - Steady State Errors

The deviation of the output of control system from desired response during steady state is known as **steady state error**. It is represented as e_{ssess}. We can find steady state error using the final value theorem as follows.

$$e_{ss}=\lim_{t\to\infty}e(t)=\lim_{s\to0}E(s)e_{ss}=\lim t\to\infty e(t)=\lim s\to0E(s)$$

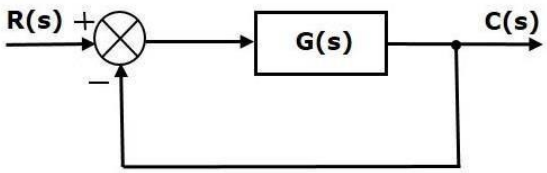
Where,

E(s) is the Laplace transform of the error signal, $e(t)e(t)$

Let us discuss how to find steady state errors for unity feedback and non-unity feedback control systems one by one.

Steady State Errors for Unity Feedback Systems

Consider the following block diagram of closed loop control system, which is having unity negative feedback.



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\Rightarrow C(s) = \frac{R(s)G(s)}{1+G(s)}$$

The output of the summing point is -

$$E(s) = R(s) - C(s)$$

Substitute $C(s)$ value in the above equation.

$$E(s) = R(s) - \frac{R(s)G(s)}{1+G(s)}$$

$$\Rightarrow E(s) = \frac{R(s) + R(s)G(s) - R(s)G(s)}{1+G(s)}$$

$$\Rightarrow E(s) = \frac{R(s)}{1+G(s)}$$

Substitute $E(s)$ value in the steady state error formula

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)}$$

Input signal	Steady state error e_{ssess}	Error constant
unit step signal	$\frac{1}{1+k_p}$11+kp	$K_p = \lim_{s \to 0} G(s)$Kp=lims→0G(s)
unit ramp signal	$\frac{1}{K_v}$1Kv	$K_v = \lim_{s \to 0} sG(s)$Kv=lims→0sG(s)

unit parabolic signal	$1K_a1K_a$	$K_a=\lim_{s\to0}s^2G(s)Ka=lims→0s2G(s)$
Input signal	**Error constant**	**Steady state error**
$r_1(t)=5u(t)r1(t)=5u(t)$	$K_p=\lim_{s\to0}G(s)=\infty Kp=lims→0G(s)=∞$	$e_{ss1}=\frac{5}{1+k_p}=0ess1=51+kp=0$
$r_2(t)=2tu(t)r2(t)=2tu(t)$	$K_v=\lim_{s\to0}sG(s)=\infty Kv=lims→0sG(s)=∞$	$e_{ss2}=\frac{2}{K_v}=0ess2=2Kv=0$
$r_3(t)=\frac{t^2}{2}u(t)r3(t)=t22u(t)$	$K_a=\lim_{s\to0}s^2G(s)=1Ka=lims→0s2G(s)=1$	$e_{ss3}=\frac{1}{k_a}=1ess3=1ka=1$

We will get the overall steady state error, by adding the above three steady state errors.

$$e_{ss}=e_{ss1}+e_{ss2}+e_{ss3}ess=ess1+ess2+ess3$$

$$\Rightarrow e_{ss}=0+0+1=1\Rightarrow ess=0+0+1=1$$

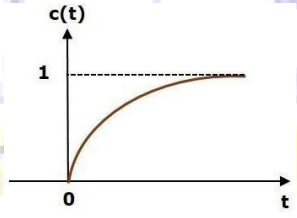
Stability

Stability is an important concept. In this chapter, let us discuss the stability of system and types of systems based on stability.

What is Stability?

A system is said to be stable, if its output is under control. Otherwise, it is said to be unstable. A **stable system** produces a bounded output for a given bounded input.

The following figure shows the response of a stable system.



This is the response of first order control system for unit step input. This response has the values between 0 and 1. So, it is bounded output. We know that the unit step signal has the value of one for all positive values of **t**including zero. So, it is bounded input. Therefore, the first order control system is stable since both the input and the output are bounded.

Types of Systemsbased on Stability

We can classify the systems based on stability as follows.

- Absolutely stable system
- Conditionally stable system
- Marginally stable system

Absolutely Stable System

If the system is stable for all the range of system component values, then it is known as the **absolutely stable system**. The open loop control system is absolutely stable if all the poles of the open loop transfer function present in left half of **'s' plane**. Similarly, the closed loop control system is absolutely stable if all the poles of the closed loop transfer function present in the left half of the 's' plane.

Conditionally Stable System

If the system is stable for a certain range of system component values, then it is known as **conditionally stable system**.

Marginally Stable System

If the system is stable by producing an output signal with constant amplitude and constant frequency of oscillations for bounded input, then it is known as **marginally stable system**. The open loop control system is marginally stable if any two poles of the open loop transfer function is present on the imaginary axis. Similarly, the closed loop control system is marginally stable if any two poles of the closed loop transfer function is present on the imaginary axis.

Control Systems - Stability Analysis

In this chapter, let us discuss the stability analysis in the **'s'** domain using the RouthHurwitz stability criterion. In this criterion, we require the characteristic equation to find the stability of the closed loop control systems.

Routh-Hurwitz Stability Criterion

Routh-Hurwitz stability criterion is having one necessary condition and one sufficient condition for stability. If any control system doesn't satisfy the necessary condition, then we can say that the control system is unstable. But, if the control syst em satisfies the necessary condition, then it may or may not be stable. So, the sufficient condition is helpful for knowing whether the control system is stable or not.

Necessary Condition for Routh-Hurwitz Stability

The necessary condition is that the coefficients of the characteristic polynomial should be positive. This implies that all the roots of the characteristic equation should have negative real parts.

Consider the characteristic equation of the order 'n' is -

$$a_0s^n+a_1s^{n-1}+a_2s^{n-2}+...+a_{n-1}s^1+a_ns^0=0a_0s^n+a_1s^{n-1}+a_2s^{n-2}+...+a_{n-1}s^1+a_ns^0=0$$

Note that, there should not be any term missing in the n^{th} order characteristic equation. This means that the n^{th} order characteristic equation should not have any coefficient that is of zero value.

Sufficient Condition for Routh-Hurwitz Stability

The sufficient condition is that all the elements of the first column of the Routh array should have the same sign. This means that all the elements of the first column of the Routh array should be either positive or negative.

Routh Array Method

If all the roots of the characteristic equation exist to the left half of the 's' plane, then the control system is stable. If at least one root of the characteristic equation exists to the right half of the 's' plane, then the control system is unstable. So, we have to find the roots of the characteristic equation to know whether the control system is stable or unstable. But, it is difficult to find the roots of the characteristic equation as order increases.

So, to overcome this problem there we have the **Routh array method**. In this method, there is no need to calculate the roots of the characteristic equation. First formulate the Routh table and find the number of the sign changes in the first column of the Routh table. The number of sign changes in the first column of the Routh table gives the number of roots of characteristic equation that exist in the right half of the 's' plane and the control system is unstable.

Follow this procedure for forming the Routh table.

- Fill the first two rows of the Routh array with the coefficients of the characteristic polynomial as mentioned in the table below. Start with the coefficient of s^n and continue up to the coefficient of s^0.
- Fill the remaining rows of the Routh array with the elements as mentioned in the table below. Continue this process till you get the first column element of **row** s^0 is a_n. Here, a_n is the coefficient of s^0 in the characteristic polynomial.

Note − If any row elements of the Routh table have some common factor, then you can divide the row elements with that factor for the simplification will be easy.

The following table shows the Routh array of the n^{th} order characteristic polynomial.

$$a_0s^n+a_1s^{n-1}+a_2s^{n-2}+...+a_{n-1}s^1+a_ns^0 a_0s^n+a_1s^{n-1}+a_2s^{n-2}+...+a_{n-1}s^1+a_ns^0$$

s^n	a_0	a_2	a_4	a_6	...	...
s^{n-1}	a_1	a_3	a_5	a_7	...	...
s^{n-2}	$b_1=\frac{a_1a_2-a_3a_0}{a_1}$	$b_2=\frac{a_1a_4-a_5a_0}{a_1}$	$b_3=\frac{a_1a_6-a_7a_0}{a_1}$	...	...	...
s^{n-3}	$c_1=\frac{b_1a_3-b_2a_1}{b_1}$	$c_2=\frac{b_1a_5-b_3a_1}{b_1}$	⋮			
⋮	⋮	⋮	⋮			
s^1	⋮	⋮				
s^0	a_n					

Example

Let us find the stability of the control system having characteristic equation,

$$s^4+3s^3+3s^2+2s+1=0 s^4+3s^3+3s^2+2s+1=0$$

Step 1 − Verify the necessary condition for the Routh-Hurwitz stability.

All the coefficients of the characteristic polynomial, $s^4+3s^3+3s^2+2s+1 s^4+3s^3+3s^2+2s+1$ are positive. So, the control system satisfies the necessary condition.

Step 2 − Form the Routh array for the given characteristic polynomial.

s^4	1	3	1
s^3	3	2	
s^2	$(3\times3)-(2\times1)3=73$	$(3\times1)-(0\times1)3=33=1$	
s^1	$(73\times2)-(1\times3)73=57$	$(73\times2)-(1\times3)73=57$	
s^0	1		

Step 3 − Verify the sufficient condition for the Routh-Hurwitz stability.

All the elements of the first column of the Routh array are positive. There is no sign change in the first column of the Routh array. So, the control system is stable.

Special Cases of Routh Array

We may come across two types of situations, while forming the Routh table. It is difficult to complete the Routh table from these two situations.

The two special cases are −

- The first element of any row of the Routh array is zero.
- All the elements of any row of the Routh array are zero.

Let us now discuss how to overcome the difficulty in these two cases, one by one.

First Element of any row of the Routh array is zero

If any row of the Routh array contains only the first element as zero and at least one of the remaining elements have non-zero value, then replace the first element with a small positive integer, ϵ. And then continue the process of completing the Routh table. Now, find the number of sign changes in the first column of the Routh table by substituting ϵ tends to zero.

Example

Let us find the stability of the control system having characteristic equation,

$$s^4+2s^3+s^2+2s+1=0$$

Step 1 − Verify the necessary condition for the Routh-Hurwitz stability.

All the coefficients of the characteristic polynomial, $s^4+2s^3+s^2+2s+1$ are positive. So, the control system satisfied the necessary condition.

Step 2 − Form the Routh array for the given characteristic polynomial.

s^4	1	1	1
s^3	~~2~~ 1	~~2~~ 1	
s^2	$(1\times1)-(1\times1)1=0$	$(1\times1)-(0\times1)1=1$	
s^1	$(1\times1)-(1\times1)1=0$	$(1\times1)-(0\times1)1=1$	

s^1			
s^0			

The row s^3 elements have 2 as the common factor. So, all these elements are divided by 2.

Special case (i) − Only the first element of row s^2 is zero. So, replace it by ϵ and continue the process of completing the Routh table.

s^4	1	1	1
s^3	1	1	
s^2	ϵ	1	
s^1	$(\epsilon \times 1) - (1 \times 1)\epsilon = \epsilon - 1$		
s^0	1		

Step 3 − Verify the sufficient condition for the Routh-Hurwitz stability.

As ϵ tends to zero, the Routh table becomes like this.

s^4	1	1	1
s^3	1	1	
s^2	0	1	
s^1	$-\infty$		
s^0	1		

There are two sign changes in the first column of Routh table. Hence, the control system is unstable.

All the Elements of any row of the Routh array are zero

In this case, follow these two steps −

- Write the auxilary equation, A(s) of the row, which is just above the row of zeros.
- Differentiate the auxiliary equation, A(s) with respect to s. Fill the row of zeros with these coefficients.

Example

Let us find the stability of the control system having characteristic equation,

$$s^5+3s^4+s^3+3s^2+s+3=0 s^5+3s^4+s^3+3s^2+s+3=0$$

Step 1 − Verify the necessary condition for the Routh-Hurwitz stability.

All the coefficients of the given characteristic polynomial are positive. So, the control system satisfied the necessary condition.

Step 2 − Form the Routh array for the given characteristic polynomial.

s^5	1	1	1
s^4	~~3~~ 1	~~3~~ 1	~~3~~ 1
s^3	$(1\times1)-(1\times1)1=0$	$(1\times1)-(1\times1)1=0$	
s^2			
s^1			
s^0			

The row s^4 elements have the common factor of 3. So, all these elements are divided by 3.

Special case (ii) − All the elements of row s^3 are zero. So, write the auxiliary equation, A(s) of the row s^4.

$$A(s)=s^4+s^2+1 \quad A(s)=s^4+s^2+1$$

Differentiate the above equation with respect to s.

$$dA(s)ds=4s^3+2s \quad \frac{dA(s)}{ds}=4s^3+2s$$

Place these coefficients in row s^3.

s^5	1	1	1
s^4	1	1	1
s^3	~~4~~ 2	~~2~~ 1	
s^2	$(2\times1)-(1\times2)=0.5$	$(2\times1)-(0\times2)=1$	
s^1	$(0.5\times1)-(1\times2)0.5=-1.50.5=-3$		
s^0	1		

Step 3 − Verify the sufficient condition for the Routh-Hurwitz stability.

There are two sign changes in the first column of Routh table. Hence, the control system is unstable.

In the Routh-Hurwitz stability criterion, we can know whether the closed loop poles are in on left half of the ‘s’ plane or on the right half of the ‘s’ plane or on an imaginary axis. So, we can’t find the nature of the control system. To overcome this limitation, there is a technique known as the root locus. We will discuss this technique in the next two chapters.

ROOT LOCUS

In the root locus diagram, we can observe the path of the closed loop poles. Hence, we can identify the nature of the control system. In this technique, we will use an open loop transfer function to know the stability of the closed loop control system.

Basics of Root Locus

The Root locus is the locus of the roots of the characteristic equation by varying system gain K from zero to infinity.

We know that, the characteristic equation of the closed loop control system is

$$1+G(s)H(s)=0$$

We can represent $G(s)H(s)$ as

$$G(s)H(s)=KN(s)D(s)$$

Where,

- K represents the multiplying factor
- N(s) represents the numerator term having (factored) n^{th} order polynomial of 's'.
- D(s) represents the denominator term having (factored) m^{th} order polynomial of 's'.

Substitute, $G(s)H(s)$ value in the characteristic equation.

$$1+kN(s)D(s)=0$$

$$\Rightarrow D(s)+KN(s)=0$$

Case 1 – K = 0

If $K=0$, then $D(s)=0$.

That means, the closed loop poles are equal to open loop poles when K is zero.

Case 2 – K = ∞

Re-write the above characteristic equation as

$$K(1K+N(s)D(s))=0\Rightarrow 1K+N(s)D(s)=0$$

Substitute, $K=\infty$ in the above equation.

$$1\infty+N(s)D(s)=0\Rightarrow N(s)D(s)=0\Rightarrow N(s)=0$$

If $K=\infty$, then $N(s)=0$. It means the closed loop poles are equal to the open loop zeros when K is infinity.

From above two cases, we can conclude that the root locus branches start at open loop poles and end at open loop zeros.

Angle Condition and Magnitude Condition

The points on the root locus branches satisfy the angle condition. So, the angle condition is used to know whether the point exist on root locus branch or not. We can find the value of K for the points on the root locus branches by using magnitude condition. So, we can use the magnitude condition for the points, and this satisfies the angle condition.

Characteristic equation of closed loop control system is

$$1+G(s)H(s)=0 1+G(s)H(s)=0$$

$$\Rightarrow G(s)H(s)=-1+j0 \Rightarrow G(s)H(s)=-1+j0$$

The **phase angle** of $G(s)H(s) G(s)H(s)$ is

$$\angle G(s)H(s)=\tan^{-1}(0-1)=(2n+1)\pi \angle G(s)H(s)=\tan^{-1}(0-1)=(2n+1)\pi$$

The **angle condition** is the point at which the angle of the open loop transfer function is an odd multiple of 180^0.

Magnitude of $G(s)H(s) G(s)H(s)$ is -

$$|G(s)H(s)|=\sqrt{(-1)^2+0^2}=1 |G(s)H(s)|=\sqrt{(-1)^2+0^2}=1$$

The magnitude condition is that the point (which satisfied the angle condition) at which the magnitude of the open loop transfer function is one.

Construction of Root Locus

The **root locus** is a graphical representation in s-domain and it is symmetrical about the real axis. Because the open loop poles and zeros exist in the s-domain having the values either as real or as complex conjugate pairs. In this chapter, let us discuss how to construct (draw) the root locus.

Rules for Construction of Root Locus

Follow these rules for constructing a root locus.

Rule 1 − Locate the open loop poles and zeros in the 's' plane.

Rule 2 − Find the number of root locus branches.

We know that the root locus branches start at the open loop poles and end at open loop zeros. So, the number of root locus branches **N** is equal to the number of finite open loop poles **P** or the number of finite open loop zeros **Z**, whichever is greater.

Mathematically, we can write the number of root locus branches **N** as

$$N=P N=P \text{ if } P \geq Z P \geq Z$$
$$N=Z N=Z \text{ if } P<Z P<Z$$

Rule 3 − Identify and draw the **real axis root locus branches**.

If the angle of the open loop transfer function at a point is an odd multiple of 180^0, then that point is on the root locus. If odd number of the open loop poles and zeros exist to the left side of a point on the real axis, then that point is on the root lo cus branch. Therefore, the branch of points which satisfies this condition is the real axis of the root locus branch.

Rule 4 − Find the centroid and the angle of asymptotes.

- If $P=Z P=Z$, then all the root locus branches start at finite open loop poles and end at finite open loop zeros.
- If $P>Z P>Z$, then $Z Z$ number of root locus branches start at finite open loop poles and end at finite open loop zeros and $P-Z P-Z$ number of root locus branches start at finite open loop poles and end at infinite open loop zeros.
- If $P<Z P<Z$, then P number of root locus branches start at finite open loop poles and end at finite open loop zeros and $Z-P Z-P$ number of root locus branches start at infinite open loop poles and end at finite open loop zeros.

So, some of the root locus branches approach infinity, when $P \neq Z P \neq Z$. Asymptotes give the direction of these root locus branches. The intersection point of asymptotes on the real axis is known as **centroid**.

We can calculate the **centroid α** by using this formula,

$$\alpha=\frac{\sum Real part of finite open loop poles-\sum Real part of finite open loop zeros}{P-Z}$$

The formula for the angle of **asymptotes θ** is

$$\theta=\frac{(2q+1)180^0}{P-Z}$$

Where,

$$q=0,1,2,....,(P-Z)-1$$

Rule 5 − Find the intersection points of root locus branches with an imaginary axis.

We can calculate the point at which the root locus branch intersects the imaginary axis and the value of **K** at that point by using the Routh array method and special **case (ii)**.

- If all elements of any row of the Routh array are zero, then the root locus branch intersects the imaginary axis and vice-versa.
- Identify the row in such a way that if we make the first element as zero, then the elements of the entire row are zero. Find the value of **K** for this combination.
- Substitute this **K** value in the auxiliary equation. You will get the intersection point of the root locus branch with an imaginary axis.

Rule 6 − Find Break-away and Break-in points.

- If there exists a real axis root locus branch between two open loop poles, then there will be a **break-away point** in between these two open loop poles.
- If there exists a real axis root locus branch between two open loop zeros, then there will be a **break-in point** in between these two open loop zeros.

Note − Break-away and break-in points exist only on the real axis root locus branches.

Follow these steps to find break-away and break-in points.

- Write K in terms of s from the characteristic equation $1+G(s)H(s)=0$.
- Differentiate K with respect to s and make it equal to zero. Substitute these values of s in the above equation.
- The values of s for which the K value is positive are the **break points**.

Rule 7 − Find the angle of departure and the angle of arrival.

The Angle of departure and the angle of arrival can be calculated at complex conjugate open loop poles and complex conjugate open loop zeros respectively.

The formula for the **angle of departure** ϕ_d is

$$\phi_d=180^0-\phi$$

The formula for the **angle of arrival** ϕ_a is

$$\phi_a=180^0+\phi$$

Where,

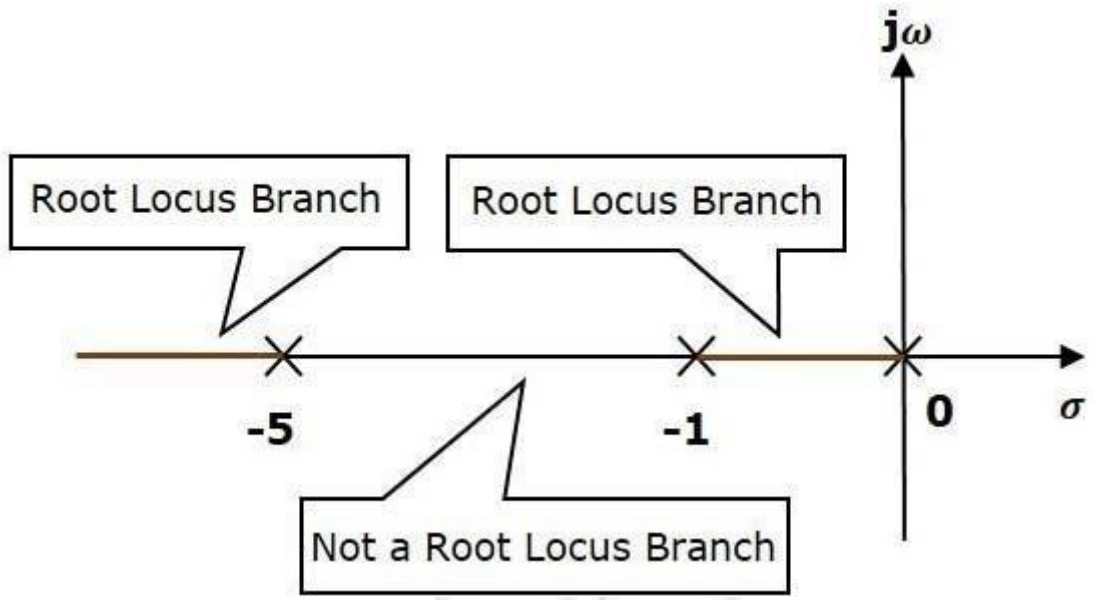
$$\phi=\sum\phi P-\sum\phi Z\phi=\sum\phi P-\sum\phi Z$$

Example

Let us now draw the root locus of the control system having open loop transfer function, $G(s)H(s)=Ks(s+1)(s+5)G(s)H(s)=Ks(s+1)(s+5)$

Step 1 − The given open loop transfer function has three poles at $s=0,s=-1s=0,s=-1$ and $s=-5s=-5$. It doesn't have any zero. Therefore, the number of root locus branches is equal to the number of poles of the open loop transfer function.

$$N=P=3N=P=3$$



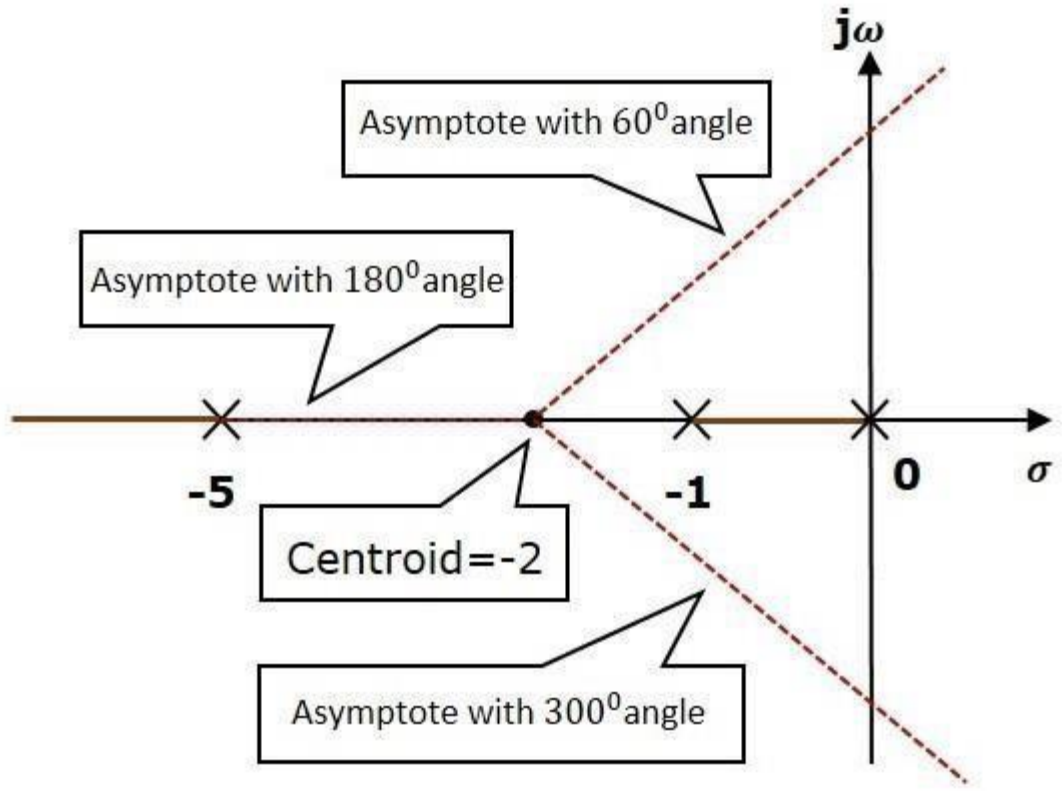
The three poles are located are shown in the above figure. The line segment between $s=-1s=-1$ and $s=0s=0$ is one branch of root locus on real axis. And the other branch of the root locus on the real axis is the line segment to the left of $s=-5s=-5$.

Step 2 − We will get the values of the centroid and the angle of asymptotes by using the given formulae.

Centroid $\alpha=-2\alpha=-2$

The angle of asymptotes are $\theta=60_0,180_0\theta=600,1800$ and 300_003000.

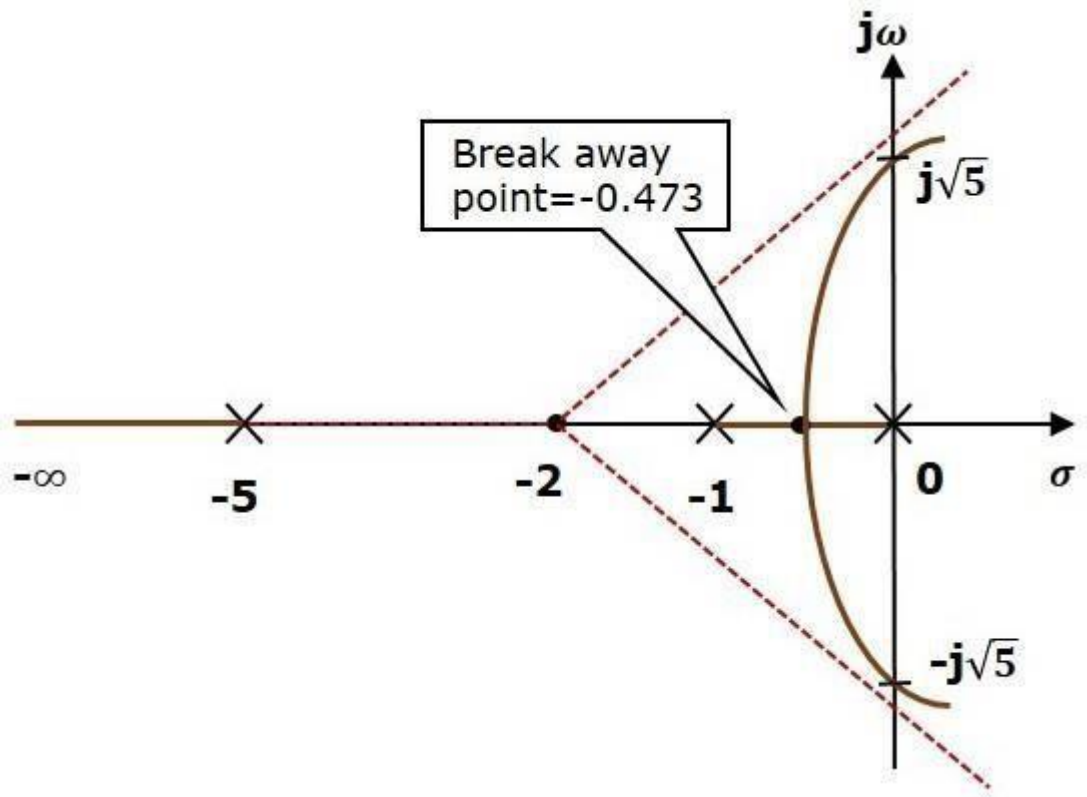
The centroid and three asymptotes are shown in the following figure.



Step 3 − Since two asymptotes have the angles of 60^{0}600 and 300^{0}3000, two root locus branches intersect the imaginary axis. By using the Routh array method and special case(ii), the root locus branches intersects the imaginary axis at j5–√j5 and −j5–√−j5.

There will be one break-away point on the real axis root locus branch between the poles s=−1s=−1 and s=0s=0. By following the procedure given for the calculation of break-away point, we will get it as s=−0.473s=−0.473.

The root locus diagram for the given control system is shown in the following figure.



In this way, you can draw the root locus diagram of any control system and observe the movement of poles of the closed loop transfer function.

From the root locus diagrams, we can know the range of K values for different types of damping.

Effects of Adding Open Loop Poles and Zeros on Root Locus

The root locus can be shifted in **'s' plane** by adding the open loop poles and the open loop zeros.

- If we include a pole in the open loop transfer function, then some of root locus branches will move towards right half of 's' plane. Because of this, the damping ratio $\delta\delta$ decreases. Which implies, damped frequency $\omega d\omega d$ increases and the time domain specifications like delay time $tdtd$, rise time $trtr$ and peak time $tptp$ decrease. But, it effects the system stability.
- If we include a zero in the open loop transfer function, then some of root locus branches will move towards left half of 's' plane. So, it will increase the control system stability. In this case, the damping ratio $\delta\delta$increases. Which implies, damped frequency $\omega d\omega d$ decreases and the time domain specifications like delay time $tdtd$, rise time $trtr$ and peak time $tptp$ increase.

So, based on the requirement, we can include (add) the open loop poles or zeros to the transfer function.